Estimating Coupling Strength in Kuramoto Coupled Oscillator Systems Using SINDy

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Abstract

Synchronization is a spontaneous event in which a system of oscillators lock to a common frequency. T'hese oscillators are not necessarily pieces of machinery, but could also be arrays of lasers, crickets that chirp in unison, or fireflies that flash at the same time again and again. Whatever the case may be, many mathematicians, like Steven Strogatz, have dedicated time and energy to study synchronization in an effort to understand and explain it. In this paper, we will attempt to do the same. First we will begin by looking back and reviewing some of the most influential literature surrounding synchronization, specifically, to work of Yoshiki Kuramoto, and his model. Then, we will utilize the Sparse Identification of Nonlinear Dynamics (SINDy) to simulate the Kuramoto model with different parameters in order to determine how well coupling strengths between oscillators in a variety of oscillator systems can be detected. Finally, we will deliver our results, analyze them, and compare them to work that has already been done using the Kuramoto model.

Introduction

In mathematics, the term synchronization refers to how a population, most often oscillators, begin rotating or working in unison. Much of the work on synchronization has been focused on collective synchronization specifically. Steven Strogatz described collective synchronization as the phenomenon "in which an enormous system of oscillators spontaneously locks to a common frequency"(5). This field of study, has been contributed to by a score of mathematicians since the mid-1900's. Some of the most fruitful work began with Winfree, but it was Yoshiki Kuramoto who really put synchronization analysis on a firmer foundation.

Around 1975, Kuramoto began working on collective synchronization, starting with Winfree's model. Kuramoto strove to simplify, or improve, his understanding of the variables involved. He chose to follow the mean-field case, which would be the most useful for his purposes. Using this case and substitution, Kuramoto determined this governing equation:

$$\theta_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \quad \text{where } i = 1, 2, 3, \ldots, N.$$

The rate of change of a given oscillator's phase over time $\frac{d\theta_i}{dt}$ is equal to its natural frequency ω_i plus the coupling strength K divided by the number of oscillators N, multiplied by $\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$ of the sine of the phase difference between every other oscillator j and the oscillator in question i.

In addition to this model, we will use the order parameter, denoted by r, to get a single measure of the degree of synchronization of the system. When r=1, the system is fully synchronized. When r = 0, that means the system is fully not synchronized. Below is a plot showing the order parameter in a system of 15 oscillators with a coupling strength of 3 over 2.5 seconds.

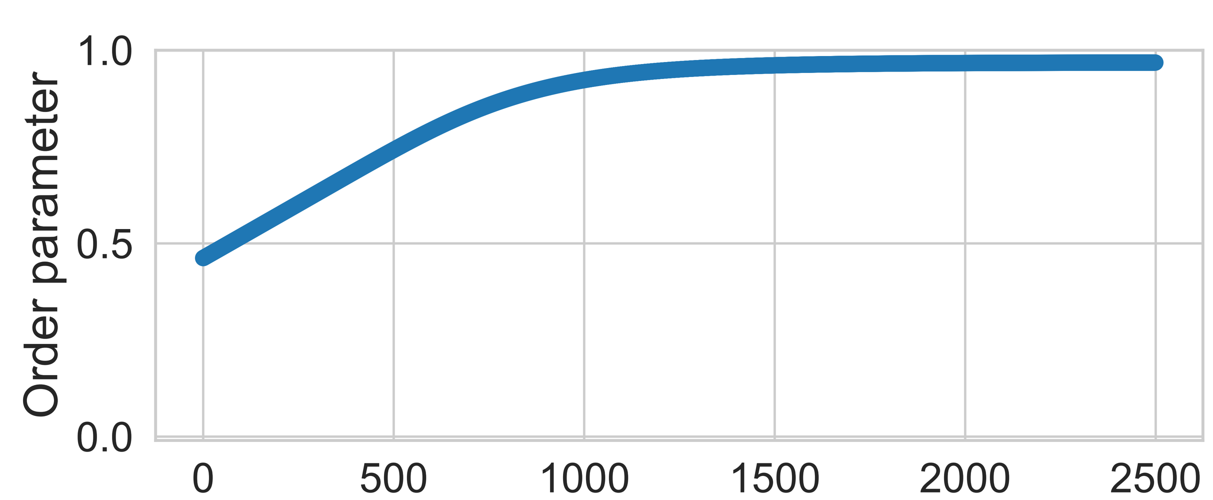

Figure 1. Time Evolution of the Order Parameter

Synchronization has a vast range of applications: oscillatory patterns are found in the brain, synchronization is used for AC generators for power grid stability, coupled laser arrays rely on synchronization to be consistent, firefly oscillations (flashes) synchronize over time, cardiac cells rely on synchronization to avoid issues. Enhancing understanding of the synchronization of coupled oscillators and the uses of the Kuramoto model can lead to new insights and discoveries in these promising fields.

This project aims to figure out if synchronization can be modeled and explained using various data-driven methods. The large set of coupled oscillators will be simulated using the Kuramoto model, which is a set of nonlinear ordinary differential equations. They're nonlinear because they involve the sine function and the only variable that is differentiated is time. Each equation in the set represents the rate of change of an oscillator's phase. The equations are coupled via θ_i and θ_j. The θ_i oscillator is affected by all other oscillators θ_j. This is modeled by $\sin(\theta_j - \theta_i)$ which captures the phase difference. They affect each other, θ_i affects θ_j and θ_j affects θ_i, the interaction is bidirectional. The sine function is used because it effectively represents how far apart two angles are. The combined effect of all oscillators θ_j on an individual oscillator θ_i is modeled by $\sum_{j=1}^{N}$.

We will simulate this model in Python using the Kuramoto library and then analyze the output data to figure out how the number of oscillators N and the distribution of natural frequencies ω affect synchronization.

Methods

Sparse Identification of Nonlinear Dynamics (SINDy) is an algorithm that is used to capture governing equations of dynamical systems from data. The SINDy algorithm is given a selection of snapshots and their corresponding time derivatives and then performs a regression using a library of nonlinear candidate functions to find governing equations for the dynamical system. The key idea behind SINDy is that most dynamical systems utilize only a sparse amount of nonlinear equations from the library of potentially relevant nonlinear functions (1).

We will simulate data using the Kuramoto model and use SINDy to identify the coupling strength between the coupled oscillators in the simulated data. First, we will explore the general dynamics of the system by varying key variables in the Kuramoto model like the coupling strength, the natural frequency distribution, the number of oscillators, the initial phase distribution, and the noise level. This exploration will help us to understand how these variables impact synchronization and which variables play a critical role in the system's dynamics. This will be done primarily by evaluating the behavior of the order parameter over time to determine if the system achieves synchronization, and if so, how quickly it synchronizes. We will pay attention to the order parameter's progression towards 1, full synchronization, or the order parameter's stabilization at a lower value, partial synchronization. Below is a plot of the Kuramoto output, showing the evolution of the same system of 15 oscillators over time.

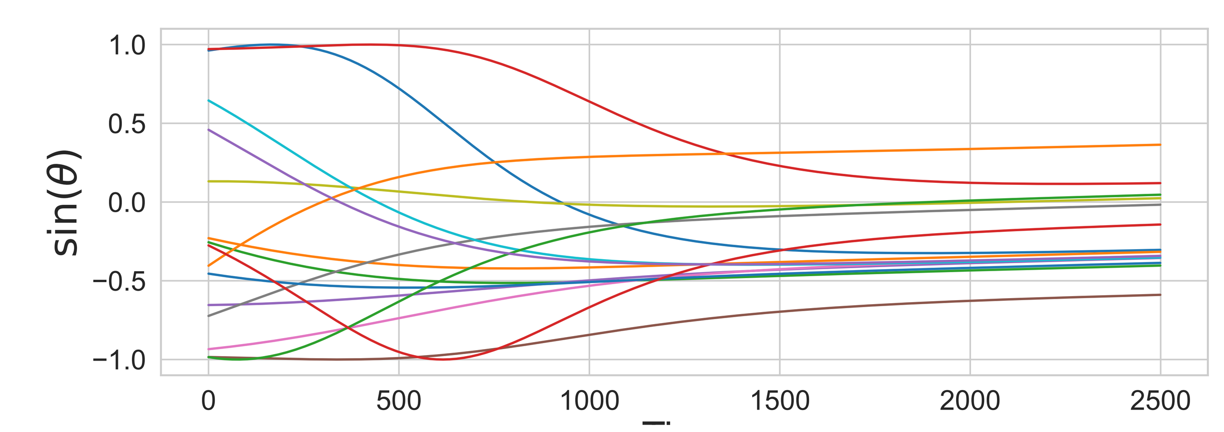

Figure 2. Activity Plot of Oscillator Phases Over Time

After establishing a basic understanding of the system's dynamics, we will use the PySINDy Python package to investigate SINDy's capacity to identify the coupling strength between oscillators. This will be done by examining phase differences using auxilary equations. A system of three coupled oscillators will have three pairwise equations:

$$x = \theta_3 - \theta_1,$$
$$y = \theta_1 - \theta_2,$$
$$z = \theta_2 - \theta_3$$

The Python code is scalable to handle any number of oscillators and generate all necessary pairwise equations. The number of formulas is $\binom{n}{2}$ for the number of oscillators n. This analysis will provide insights into the SINDy algorithm's effectiveness with oscillatory systems such as the Kuramoto model. The extended description of the script can be found in the paper. Next is a plot of the pairwise difference derivatives over time. With 15 oscillators, there are 105 pairwise differences being calculated. We can still see a general convergence at a similar point in time as the order parameter plot and the activity plot.

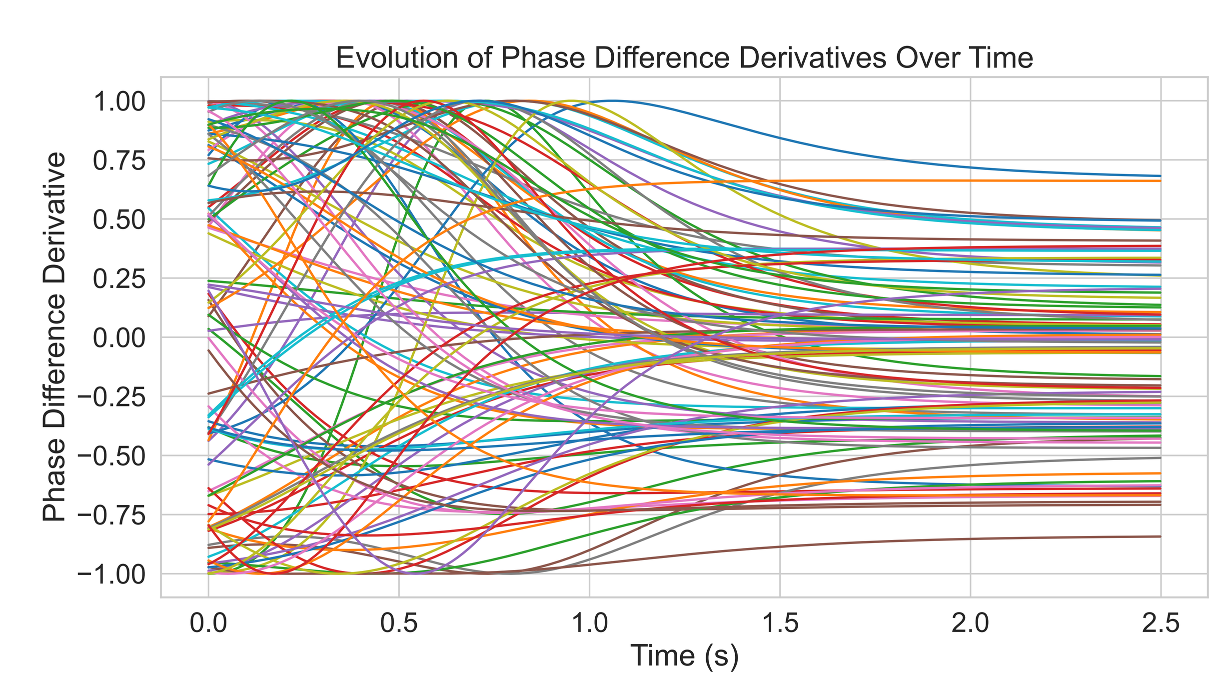

Figure 3. Pairwise Phase Differences Among 15 Oscillators

Results and Analysis

The coefficients returned from SINDy appear to be consistently directly related to the initial coupling value. There appears to be a scaling factor for the initial coefficient as the number of oscillators increases. With only two oscillators the coefficients are very close to the original coupling strength values.

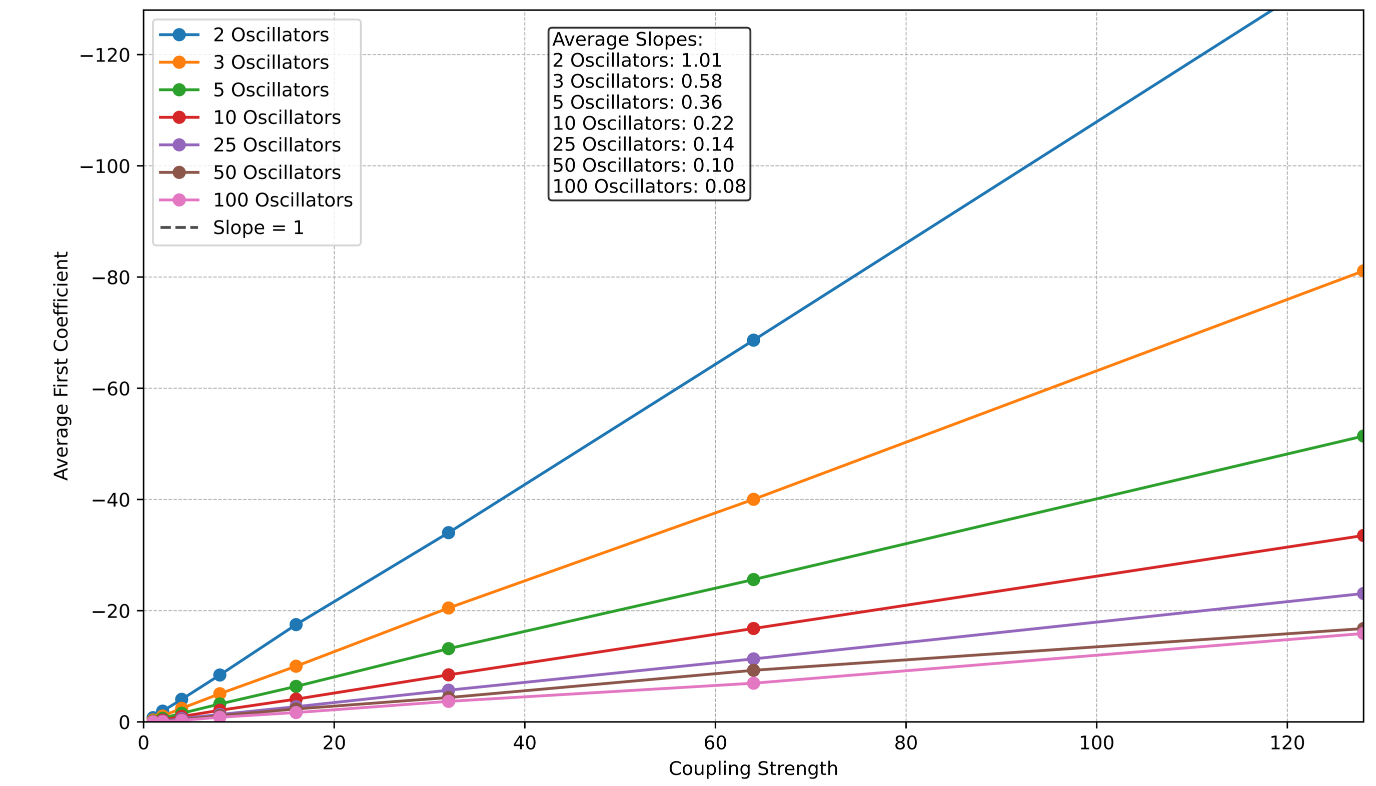

Figure 4. Average First Coefficient vs. Coupling Strength for Different Numbers of Oscillators

As shown in the above figure, a plot of our results, with 2 oscillators, the coefficient returned by SINDy is very close to the original coupling strength value. The magnitude of the factor of difference, visualized as the slope in this plot, is about 1 for 2 oscillator systems that consistently synchronize, which happens when the coupling strength is 2 or greater. As the number of oscillators increases, we can see consistency in the slopes of those systems that consistently synchronize, but the factor decreases at some scale.

Discussion

One result that becomes evident from our work is the change in the average first coefficient, for each set of n oscillators, as the coupling strength varies. Like Duncan Watts and Steven Strogatz, we wanted to see what outcomes would be visible if we kept the number of oscillators the same, but changed the coupling strength (k)(6). Watts and Strogatz applied this tactic to 'small-world' networks in 1998. Specifically, they arranged 20 vertices in a ring, and wired connections between each vertices and its four closest neighbors. In this initial step, the probability p was equal to 0 as the ring was unchanged and considered "completely regular" (6). and hanging p meant rewiring the connections. When the probability equaled one, the network was considered completely random since some vertices had two connections whereas others had four or more. Between p=0 and p=1, Watts and Strogatz found that the graph truly became a 'small-world' network as the average characteristic path length decreased. Smaller path lengths lead to enhanced signal-propagation speed, and eventually synchronization. As Nguyen, Honda, Nakamura, Sano, and several others would go on to show, Kuramoto's model was useful in more than one type of dynamical data.

A more recent article documents their application of the Kuramoto model to graph neural networks (4). Similar to the ring model Watts and Strogatz employed, graph neural networks are layered arrangements of nodes which are connected by edges. These data systems aggregate information via message passing between nodes along the edges of each layer. One limitation of graph neural networks is a problem called over-smoothing (4) which occurs when a specific graph neural network repeatedly aggregates information from neighboring nodes. As this happens, the representations of nodes from different classes can become indistinguishable. Nguyen and all sought to tackle this problem by utilizing the Kuramoto model because, "there is a connection between synchronization and the over-smoothing phenomenon in GNNs. Both involve a collective behavior of the nodes in the network, where nodes become more similar to each other over time" (4).

Conclusion

In this paper we defined the phenomenon that is synchronization, and reviewed one of the most influential models for observing and analyzing rates of synchronization. We also set up our experiment using the Kuramoto model for data simulation and, using SINDy, were able to reliably estimate the original coupling strength in Kuramoto just from the output data. In our experiment, the dynamics in each system with a set number of oscillators appear to be consistent under the previously outlined conditions. This suggests that real-world data could be analyzed via SINDy to return to a Kuramoto model that could be further tested and adjusted.