THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Trajectory Inference in Wasserstein Space

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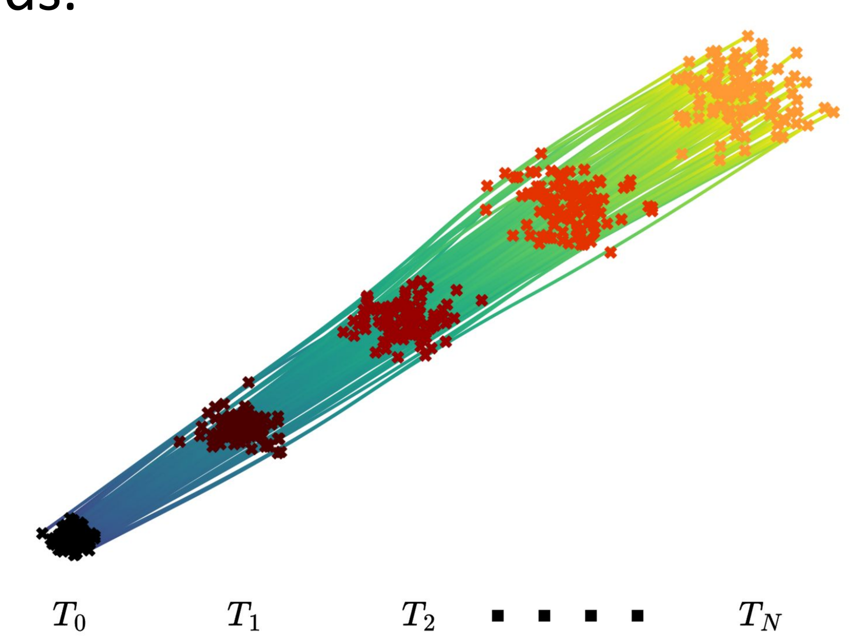
Trajectory Inference

Reconstructing continuous trajectories between point clouds.



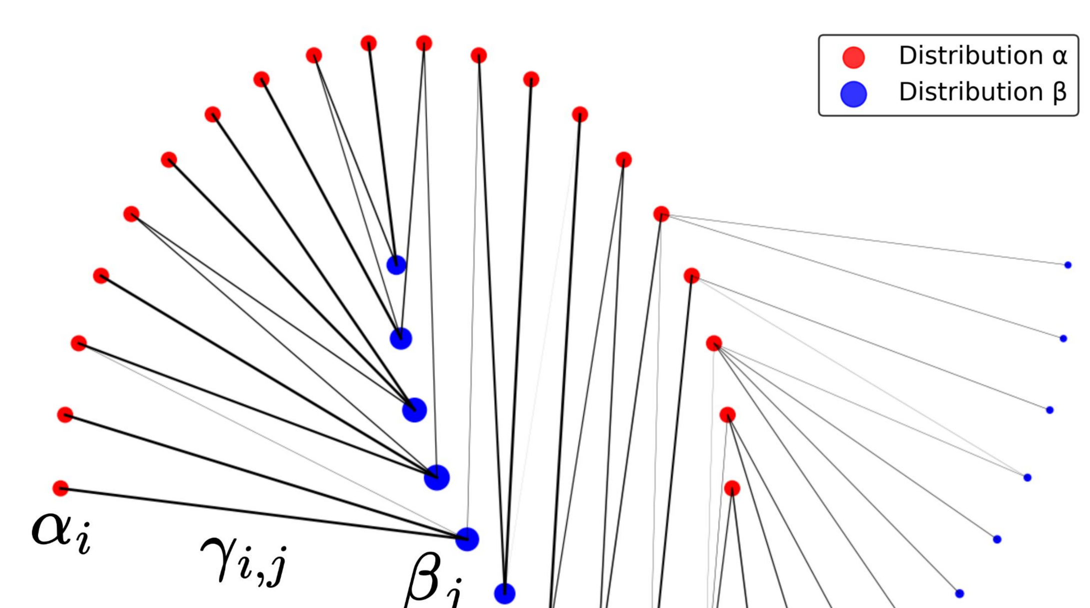
Given an ordered sequence of probability measures $\{\mu_{t_j}\}$ at times $t_0 < t_1 < \cdots < t_T$, we aim to find a curve $\nu : \mathbb{R} \to \mathcal{P}_p(\mathbb{R}^d)$ such that:

Interpolation: $\nu(t_j) = \mu_{t_j}$ for all j, or

Approximation: $\nu(t_j) \approx \mu_{t_j}$, i.e. $\mathcal{P}_p(\nu(t_j), \mu_{t_j}) < \varepsilon$.

Optimal Transport (OT)

OT seeks to move mass of distribution $\boldsymbol{\alpha}$ to distribution $\boldsymbol{\beta}$ in a way that minimizes work (or cost).



Exact OT: Linear Program

$$(W_p)^p = \min_{\gamma} \langle \gamma, \mathbf{C} \rangle_F = \sum_{i,j} \gamma_{i,j} C_{i,j}$$

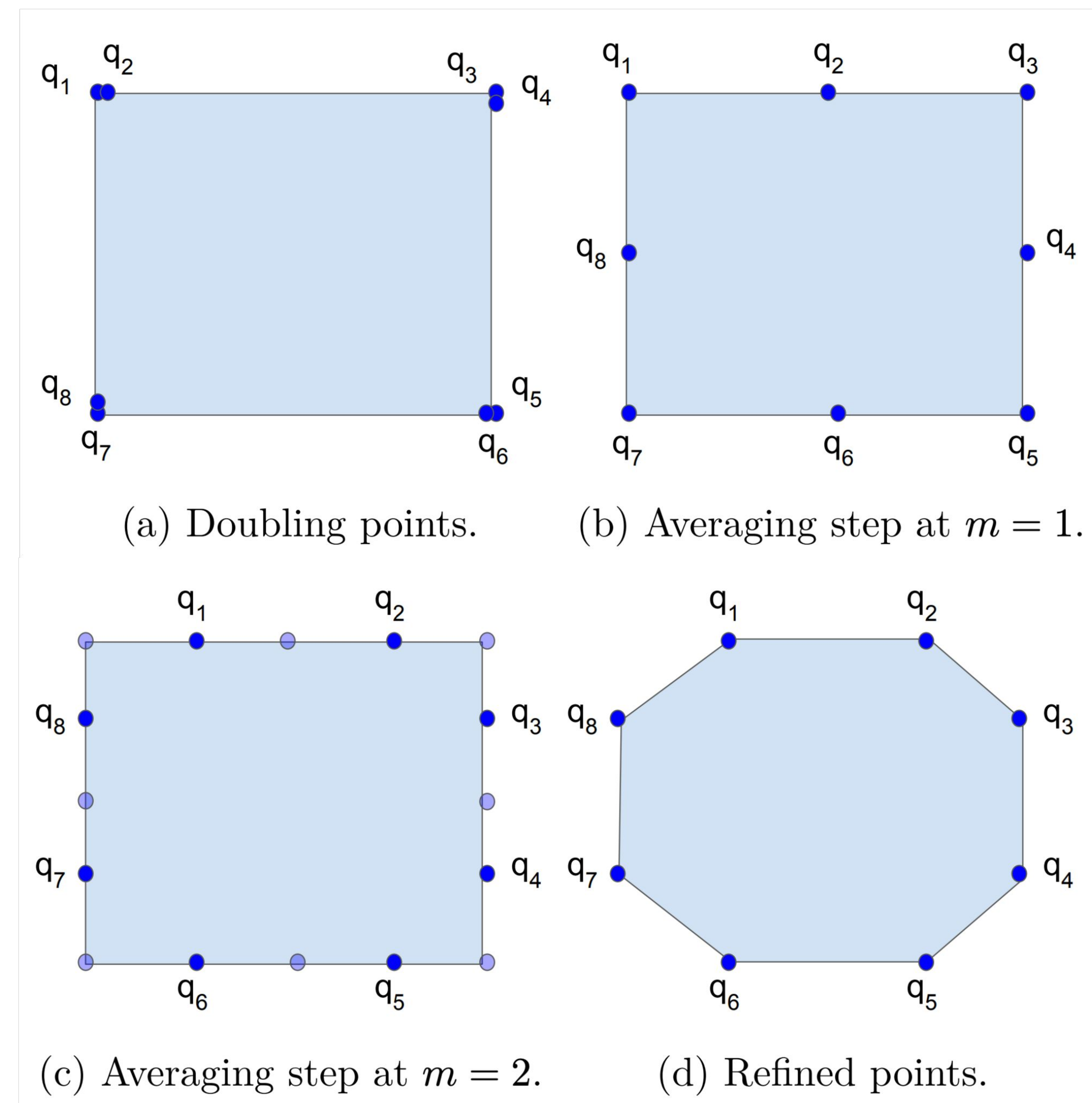
s.t. $\gamma \mathbf{1} = \alpha$, $\quad \gamma^T \mathbf{1} = \beta$, $\quad \gamma \geq 0$

α = source
β = target
C = cost matrix

Where γ is the optimal coupling between α and β

Lane Riesenfeld Algorithm

The Lane-Riesenfeld algorithm generates smooth curves through an iterative process involving **doubling** and **refinement**, which increase the number of control points, and **smoothing** via *averaging*. Repeating these steps leads to the convergence to a smooth limit curve.



(a) Doubling points. (b) Averaging step at $m = 1$.

(c) Averaging step at $m = 2$. (d) Refined points.

OT Averaging

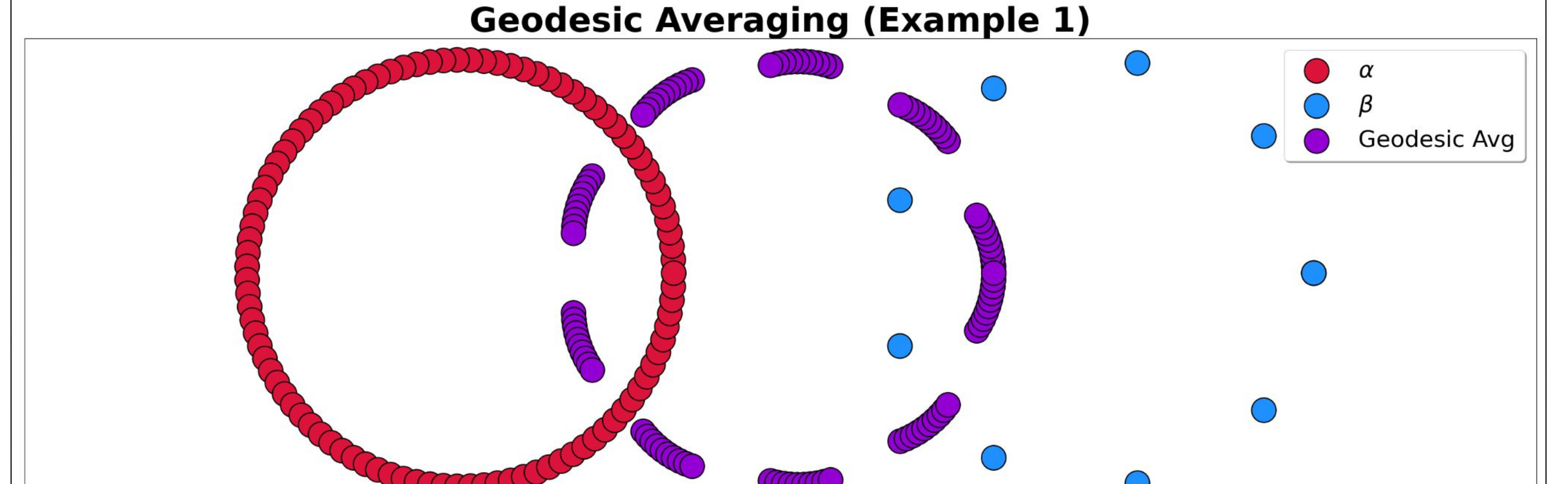
$$\text{mean}(\alpha, \beta) = \sum_{i=1}^{n_\alpha} \sum_{k=1}^{n_\beta} \gamma_{ik} \delta_{\frac{1}{2}x_{i,\alpha} + \frac{1}{2}x_{k,\beta}}$$

$$\alpha = \sum_{i=1}^{n_\alpha} \alpha_i \delta_{x_i} \qquad \sum_{i=1}^{n_\alpha} \alpha_i = 1$$

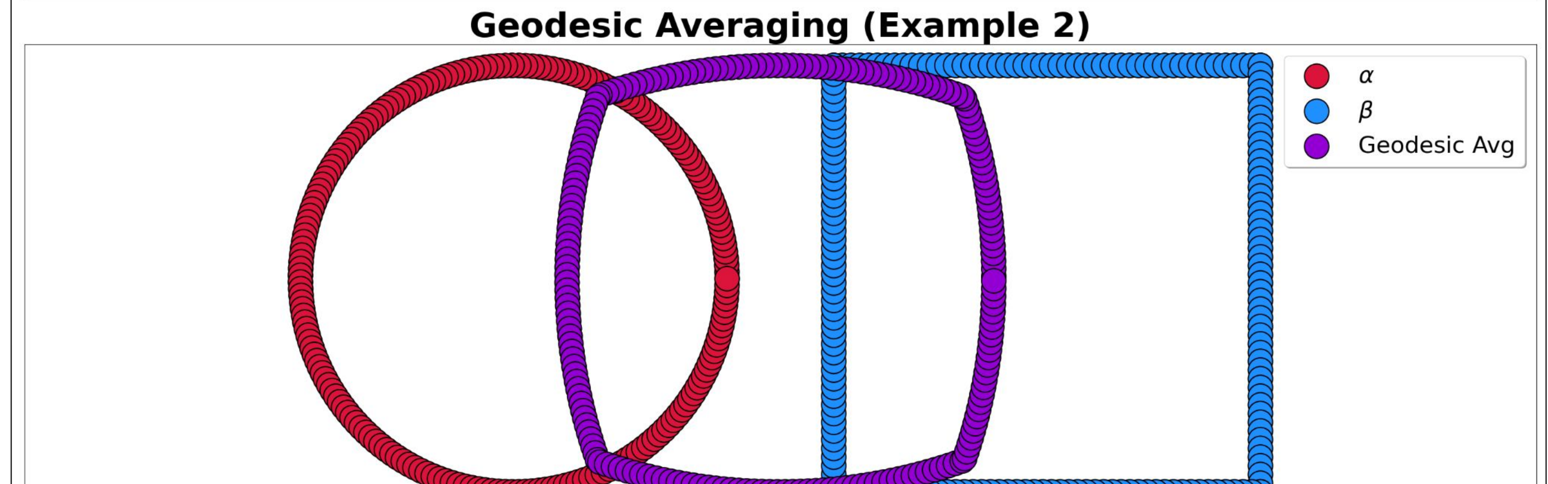
$$\beta = \sum_{j=1}^{n_\beta} \beta_j \delta_{x_j} \qquad \sum_{j=1}^{n_\beta} \beta_j = 1$$

Here γ is the optimal coupling between α and β

Geodesic Averaging (Example 1)



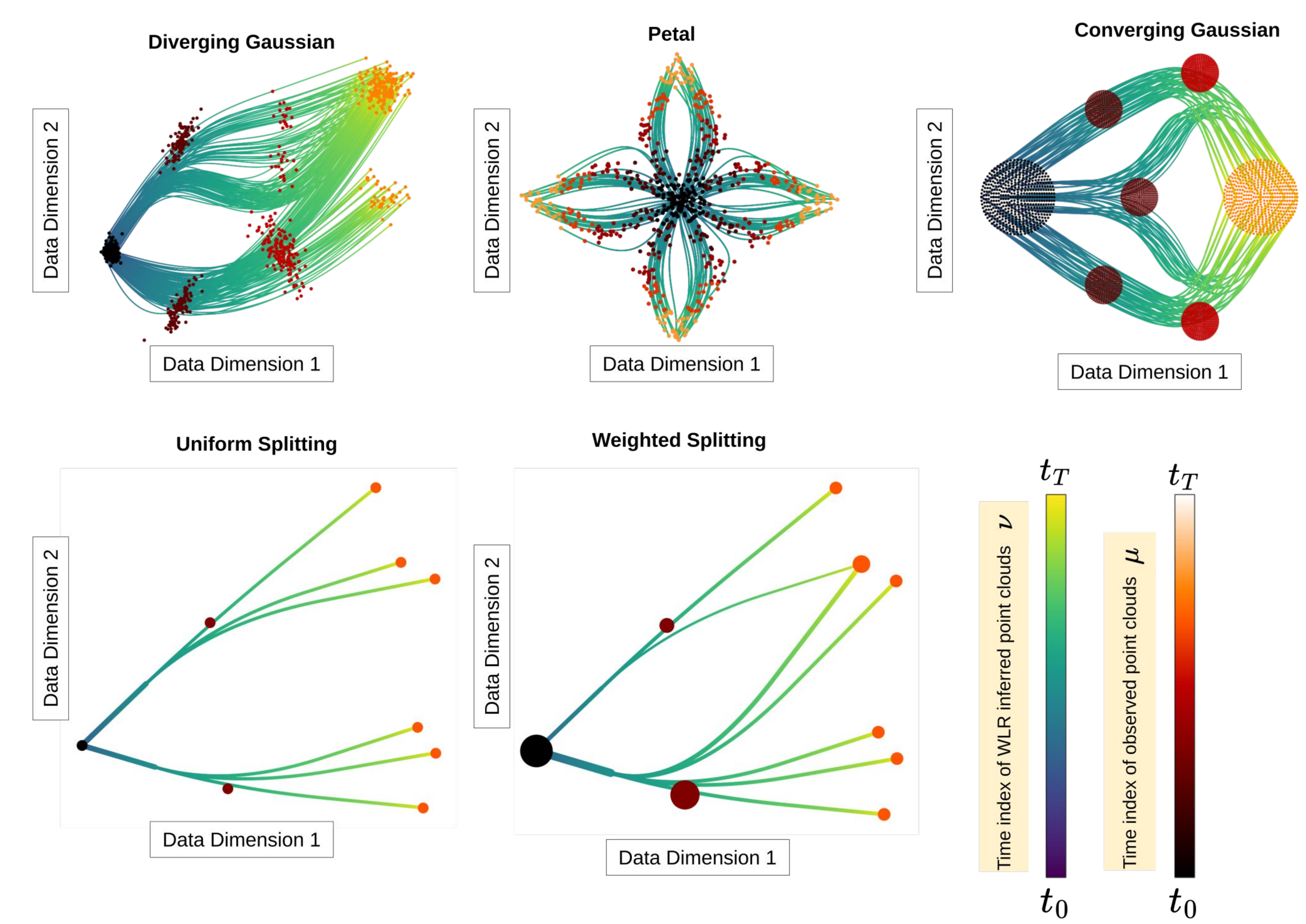
Geodesic Averaging (Example 2)



Wasserstein Lane Riesenfeld Algorithm

procedure WASSERSTEIN-LANERIESENFELD($[\mu_{t_j}]_{j=0}^T, R, M$)
  Input: Point *clouds* to be refined $[\mu_{t_j}]_{j=0}^T$
  Input: Refinement Level $R \in \mathbb{Z}_+$
  Input: Degree M of B-Splines to be approximated
  $\nu^{(M)} \leftarrow [\mu_{t_j}]_{j=0}^T$ ▷ Initializing point clouds to be doubled
  for $r = 1$ **to** R **do**
    for $j = 0$ **to** $|\nu^{(M)}|$ **do** ▷ Doubling point clouds
      $\nu_{2j}^{(0)} \leftarrow \nu_j^{(M)}$
      $\nu_{2j+1}^{(0)} \leftarrow \nu_j^{(M)}$
    end for
    $\nu^{(0)} \leftarrow [\underbrace{\nu_0^{(0)}, \ldots, \nu_0^{(0)}}_{m \text{ times}}, \nu_1^{(0)}, \ldots, \nu_{T-1}^{(0)}, \underbrace{\nu_T^{(0)}, \ldots, \nu_T^{(0)}}_{m \text{ times}}]$
    for $m = 1$ **to** M **do**
      for $j = 0$ **to** $|\nu^{(m-1)}|$ **do**
        $\nu_j^{(m)} \leftarrow \text{OT-av}(\nu_j^{(m-1)}, \nu_{j+1}^{(m-1)}, \frac{1}{2})$ ▷ Repeated OT averaging
      end for
    end for

  return Refined point clouds $\nu^{(M)}$. $|\nu^{(M)}| = 2^R(T + M - 1) + 2 - M$



Acknowledgements

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